Continuous time random walks on moving fluids

Albert Compte*
Departament de Física, Física Estadística, Universitat Autònoma de Barcelona, E-08193 Bellaterra, Catalonia, Spain
(Received 12 February 1997)

The scheme of the continuous time random walk (CTRW) is generalized to include the possibility of a moving background. It is shown that this generalization reproduces in the macroscopic limit the usual diffusion-advection equation and the properties of standard diffusion in a shear flow. The new formalism is then used to derive the corresponding macroscopic equation for CTRW's with infinite mean squared step length and with infinite mean waiting time in a moving fluid. For these two CTRW's we finally include an analysis of the dispersion in three different two-dimensional linear shear flows. [S1063-651X(97)15506-4]

PACS number(s): 05.40.+j, 05.60.+w, 02.50.−r, 47.15.−x

I. INTRODUCTION

Diffusion in a velocity field is an important issue of diffusion theory that is relevant in many industrial (mixing of fluids, chemical reactions, etc.), technological (electronic transport, electrophoresis, etc.), and environmental (sedimentation, spreading of pollutants, etc.) situations besides many basic topics (Taylor diffusion, Rayleigh-Bénard, etc.). This subject has been profusely studied from a variety of points of view: diffusion-advection equation [1,2], Langevin equation [1], path integrals [3], and numerical simulations [1] and here we shall develop a new stochastic method to analyze it, which can also be applied when the governing statistics is not Gaussian but stable (Lévy statistics). This method originates as a generalization of the continuous time random walk scheme [4–7].

Continuous time random walks (CTRW's) have proved the natural way to incorporate stable distributions or Lévy distributions and anomalous scaling into the realm of random walks as Lévy flights. These Lévy flights have been applied with very good agreement to many physical experiments: diffusion of carriers in an amorphous photoconductor [8], anomalous diffusion in rotating fluids [9], turbulent diffusion [10], transport in turbulent plasma [11], vortex dynamics [12], and others [13,14]. In all of these cited examples diffusion is usually associated with a drift in the underlying medium, be it a fluid or an electric field. This fact indicates the importance of a study of the coupling of Lévy flights with a velocity field. Some few works [15–17] have approached this question through heuristic generalizations of the diffusion-advection equation but up to now it has not yet been justified that these generalizations correspond to Lévy flights evolving in nonhomogeneous flows. This is the question which we shall address in this paper, not through the diffusion-advection equation as in [15–17] but following a stochastic approach. To this aim we will need to generalize the scheme of CTRW by allowing the step length distribution function to depend on the starting point of the jump. In Sec. II we develop this generalization and test it with Brownian diffusion, we prove that the results concerning the diffusion-advection equation and the mean square displacement in three kinds of shear flows are coincident with those of standard diffusion theory. Section III contains the application of this method to Lévy flights of infinite mean waiting time and infinite mean square step length respectively. For this last point the theory of ordered spans [18] must be used in order to get conclusions on the rate of diffusion. Most results of this section for moving fluids are original and difficult to obtain by other methods. Finally the conclusions are exposed in Sec. IV.

II. BROWNIAN DIFFUSION

A. Diffusion-advection equation

In the continuous time random walk scheme [4–7] the quantity which defines the motion is the probability distribution $\psi(\mathbf{r},\tau)$ of the random walker performing a jump of length $\mathbf{r}$ after waiting a time τ at its starting point. Its Fourier-Laplace transform is related to the integral transform of the probability density $\rho(\mathbf{x},t)$ of the walker being at time t at point $\mathbf{x}$ through the well-known relation [4,7]

$$\rho(\mathbf{k},u)=\frac{1}{u}\,\frac{1-\varphi(u)}{1-\psi(\mathbf{k},u)}, \tag{1}$$

where $\varphi(\tau)=\int d\mathbf{r}\;\psi(\mathbf{r},\tau)$ is the waiting time distribution function and by explicitly displaying the dependence on $\mathbf{k}$ and u we indicate that $\rho(\mathbf{k},u)$ is the Fourier-Laplace transform of $\rho(\mathbf{x},t)$. The relation (1), upon inversion, solves the problem of the CTRW governed by the probability distribution $\psi(\mathbf{r},\tau)$. This same quantity also provides the form of the generalized diffusion equation of the motion as [7]

$$\frac{\partial\rho(\mathbf{x},t)}{\partial t}=\int d\mathbf{x}'\int d\tau\;\;K(\mathbf{x}-\mathbf{x}',t-\tau)\rho(\mathbf{x}',\tau) \tag{2}$$

with

$$K(\mathbf{k},u)=u\,\frac{\psi(\mathbf{k},u)-\varphi(u)}{1-\varphi(u)}. \tag{3}$$

The relation between CTRW's and fractional derivatives [19] has been proved both from (1) [20,21] and from (3) [22]. In the next section we will make use of this relation to write down diffusion-advection equations for Lévy flights.

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*Electronic address: albert@telemaco.uab.es

In this article our interest will be focused on random walkers in moving fluids, that is to say, on biased CTRW's. It has already been observed [23] that for some random walk models with bias, the structure function $\lambda(\mathbf{k}) = \int d\tau\ \psi(\mathbf{k},\tau)$ has an imaginary linear dependence on $\mathbf{k}$ in the short $\mathbf{k}$ limit, which does not appear in symmetric walks. We shall argue now why this should be general and this will give us the clue for our further developments.

We start with the standard diffusion-advection equation for diffusion in a fluid moving along the x direction with constant velocity v:

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} + v\,\frac{\partial \rho(\mathbf{x},t)}{\partial x} = D\nabla^2 \rho(\mathbf{x},t). \tag{4}$$

In Fourier space this equation turns into

$$\frac{\partial \rho(\mathbf{k},t)}{\partial t} = -(ik_x v + Dk^2)\rho(\mathbf{k},t),$$

which in the light of Eqs. (2) and (3) corresponds to a step length probability distribution

$$\lambda(\mathbf{k}) = 1 - i\tau v k_x - \tau D k^2, \tag{5}$$

where we assume, as in Brownian diffusion, that $\varphi(u) \simeq 1 - \tau u$ for $u \to 0$ [4]. The term $-i\tau v k_x$ obviously comes from the advection term in Eq. (4) and therefore contains the essential information for advection at long distances, where one assumes Eq. (4) to be generally valid irrespective of the precise form for the $\psi(\mathbf{r},t)$ which defines the diffusive motion. Now we argue that this term might be understood in the frame of CTRW as a displacement in the original (in a resting fluid) step length distribution $\psi_0(\mathbf{r},t)$: in a fluid moving with constant velocity $\mathbf{v}$

$$\psi(\mathbf{r},t) = \psi_0(\mathbf{r} - \mathbf{v}\tau_a, t), \tag{6}$$

where τ_a stands for an advection time scale which in Brownian diffusion coincides with the mean waiting time at a site τ but for some Lévy flights, as we will see, it needs be different to provide a well defined macroscopic limit. Indeed, this interpretation is consistent with our previous result (5) since a Fourier transformation yields

$$\psi(\mathbf{k},t) = e^{-i\tau_a \mathbf{k}\cdot\mathbf{v}} \psi_0(\mathbf{k},t) \tag{7}$$

and in the short $\mathbf{k}$ limit (long distances) the term $-i\tau_a\mathbf{k}\cdot\mathbf{v}$ appears. We see this more explicitly now by considering a particular CTRW which leads to Brownian diffusion. We make the choice $\psi_0(\mathbf{r},t) = Ce^{-t/\tau} e^{-r^2/4\sigma^2}$ being C a normalization constant. In the Fourier-Laplace space and using (7) we get

$$\psi(\mathbf{k},u) = e^{-i\tau_a \mathbf{k}\cdot\mathbf{v}} \frac{1}{1+u\tau} e^{-\sigma^2 k^2}$$

so that, following (3), the kernel of the associated generalized diffusion equation is

$$K(\mathbf{k},u) = \frac{1}{\tau}\left(e^{-i\tau_a \mathbf{k}\cdot\mathbf{v} - \sigma^2 k^2} - 1\right).$$

If we wish to retain just the essential properties of the motion which manifest macroscopically, independent of the particular stochastic model chosen, we need to take the limits $\tau \to 0$, $\tau_a \to 0$ and $\sigma \to 0$ keeping $D = \sigma^2/\tau$ and $A = \tau_a/\tau$ constants. After taking these limits, we have

$$K(\mathbf{k},u) = -iA\mathbf{k}\cdot\mathbf{v} - Dk^2$$

and inserting this into Eq. (2) we get the standard diffusion-advection equation, provided we have $A = 1$ (this assumption does not limit the generality at all since A, being nondimensional, can be absorbed into a rescaling of t and D):

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} + \mathbf{v}\cdot\nabla\rho(\mathbf{x},t) = D\nabla^2\rho(\mathbf{x},t). \tag{8}$$

B. CTRW scheme for inhomogeneous flows

We shall now allow for a nonhomogeneous velocity field $\mathbf{v}(\mathbf{x})$. This implies that the standard scheme of CTRW must be revised to account for this inhomogeneity since now the probability ϕ of a length of step $\mathbf{r}$ with waiting time τ will crucially depend on the velocity of the fluid at the starting point of the jump $\mathbf{x}$:

$$\phi = \phi(\mathbf{r},\tau;\mathbf{x}) = \psi_0(\mathbf{r} - \tau_a \mathbf{v}(\mathbf{x}), \tau). \tag{9}$$

The CTRW scheme has to be now reformulated: if $P(\mathbf{x},t)$ is the probability density of arriving at point $\mathbf{x}$ at time t and $\rho(x,t)$ is the probability density of being at point $\mathbf{x}$ at time t, we have [7]

$$P(\mathbf{x},t) = \int d\mathbf{x}' \int_0^t dt'\ \phi(\mathbf{x}-\mathbf{x}', t-t'; \mathbf{x}')\ P(\mathbf{x}',t') + \delta(\mathbf{x})\delta(t), \tag{10}$$

$$\rho(\mathbf{x},t) = \int_0^t d\tau\ P(\mathbf{x},t-\tau)\Psi(\tau), \tag{11}$$

where we have introduced in (11) the probability $\Psi(\tau)$ of remaining at least a time τ on the spot before proceeding with another jump ($\Psi(\tau) = \int_\tau^\infty \varphi(\tau) d\tau$) and we have incorporated the initial conditions for a pulse initially concentrated at the origin in the form of the delta functions in (10). We now combine Eqs. (10) and (11) to get

$$\rho(\mathbf{x},t) = \int_0^t d\tau\ \Psi(\tau) \int d\mathbf{x}' \int_0^{t-\tau} dt'\ \phi(\mathbf{x}-\mathbf{x}', t-\tau-t'; \mathbf{x}') \times P(\mathbf{x}',t') + \delta(\mathbf{x})\Psi(t).$$

Performing now the variable change $t' = t'' - \tau$ and inverting the order of the time integrals we arrive, in the Fourier-Laplace domain, to

$$\rho(\mathbf{k},u) = \int d\mathbf{k}'\ \phi(\mathbf{k},u;\mathbf{k}-\mathbf{k}')\ \rho(\mathbf{k}',u) + \Psi(u). \tag{12}$$

Equation (12) will be the starting point for the remainder of the article and is a generalization of equation Eq. (1) for the case of CTRW's in nonresting backgrounds, where one has, according to (9),

$$\phi(\mathbf{k},u;\mathbf{k}')=\psi_0(\mathbf{k},u)\int d\mathbf{x}'\, e^{-i\mathbf{k}'\cdot\mathbf{x}'}e^{-i\tau_a\mathbf{k}\cdot\mathbf{v}(\mathbf{x}')}. \quad (13)$$

C. Linear shear flows

We shall now consider some particular instances of flows where this scheme can easily be applied. They will all be linear shear flows, i.e., flows defined by a velocity field of the form $\mathbf{v}(\mathbf{x})=\mathbf{\Omega}\cdot\mathbf{x}$, with $\mathbf{\Omega}$ a constant square matrix with inverse time dimensions. By using (12) and (13) we get the equation

$$\rho(\mathbf{k},u)=\psi_0(\mathbf{k},u)\;\rho(\mathbf{k}+\mathbf{V}_\mathbf{k},u)+\Psi(u) \quad (14)$$

with $\mathbf{V}_\mathbf{k}=\tau_a\mathbf{\Omega}^{\mathrm{T}}\cdot\mathbf{k}$. In the macroscopic limit ($\tau_a\to 0$) we observe that $\mathbf{V}_\mathbf{k}\to 0$, so that we approximate (14) for $\mathbf{V}_\mathbf{k}\simeq 0$. This yields

$$[1-\psi_0(\mathbf{k},u)]\;\rho(\mathbf{k},u)\simeq\psi_0(\mathbf{k},u)\;\mathbf{V}_\mathbf{k}\cdot\nabla_\mathbf{k}\rho(\mathbf{k},u)+\Psi(u) \quad (15)$$

and now the corresponding macroscopic diffusion-advection equation can be derived for any kind of CTRW by inverting the Fourier-Laplace transform and performing the convenient macroscopic limit. To compare with the standard results we try now Brownian diffusion, where $\psi_0(\mathbf{k},u)=(1+u\tau)^{-1}\exp(-\sigma^2\mathbf{k}^2)$ and obtain

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\mathbf{v}\rho)=D\nabla^2\rho.$$

This scheme, though, does not only show the form of the advection-diffusion equation for CTRW's but can also yield results concerning the properties of such a diffusive motion. In particular, an important quantity that characterizes diffusion is the mean square displacement, which gives an idea of the rate of diffusion of the walker. We now apply our generalized CTRW formalism to obtain this quantity for Brownian diffusion in three important two-dimensional cases: simple shear, pure rotation, and pure shear. We will thus check the validity of the model since all these properties are well known for Brownian diffusion [1].

1. Simple shear

In this case we have $\mathbf{v}(\mathbf{x})=(\omega y,0)$, or equivalently

$$\Omega=\begin{pmatrix}0 & \omega\\ 0 & 0\end{pmatrix},$$

from where $V_k=(0,\tau_a\omega k_x)$ so that Eq. (14) turns into

$$\rho(k_x,k_y,u)=\psi_0(k_x,k_y,u)\;\rho(k_x,k_y+\tau_a\omega k_x,u)+\Psi(u). \quad (16)$$

We now repeatedly iterate relation (16) to obtain after N iterations

$$\begin{aligned}\rho(k_x,k_y,u)=&\rho(k_x,k_y+N\tau_a\omega k_x,u)\\ &\times\prod_{n=0}^{N-1}\psi_0(k_x,k_y+n\tau_a\omega k_x,u)+\Psi(u)\\ &\times\sum_{n=0}^{N-2}\prod_{m=0}^{n}\psi_0(k_x,k_y+m\tau_a\omega k_x,u)\;+\Psi(u).\end{aligned} \quad (17)$$

In the limit $N\to\infty$ we have in the first summand of (17) a bounded function ρ multiplied by an infinite product of terms smaller than one; this term obviously vanishes in the limit that we are considering, so that we may write

$$\rho(k_x,k_y,u)=\Psi(u)\sum_{n=0}^{\infty}\prod_{m=0}^{n}\psi_0(k_x,k_y+m\tau_a\omega k_x,u)+\Psi(u), \quad (18)$$

which now solves the problem for any possible CTRW in a simple shear flow. A first conclusion to be drawn is that in the y direction diffusion proceeds unaffected by the shear flow as becomes clear setting $k_x=0$ (average over direction x) in (18) and performing the product and the sum indicated to get

$$\rho(0,k_y,u)=\frac{1}{u}\,\frac{1-\varphi(u)}{1-\psi_0(0,k_y,u)},$$

which turns out to be identical to the solution of a CTRW in a resting fluid (1). We therefore only expect novelties in diffusion along the x axis. For Brownian diffusion with $\psi_0(k_x,k_y,u)=\varphi(u)\exp(-\sigma^2k_x^2-\sigma^2k_y^2)$, $\varphi(u)=(1+u\tau)^{-1}$ and averaging over the y direction ($k_y=0$), we get

$$\begin{aligned}\rho(k_x,0,u)=&\Psi(u)\sum_{n=0}^{\infty}(\varphi(u)e^{-\sigma^2k_x^2})^{n+1}\\ &\times\exp\left(-\sigma^2\tau_a^2\omega^2k_x^2\frac{n(n+1)(2n+1)}{6}\right)\\ &+\Psi(u).\end{aligned}$$

We are now interested in the moments of this distribution; it is easy to see that $\langle x\rangle=0$ so the dispersion $\langle\delta x^2\rangle$ will directly be given by $\langle x^2\rangle$ and is easily computed from the Fourier transform of ρ as

$$\langle x^2\rangle=-\left.\frac{\partial^2\rho(k_x,0,u)}{\partial k_x^2}\right|_{k_x=0}.$$

We therefore have

$$\langle\delta x^2\rangle=2\Psi(u)\sigma^2\sum_{n=0}^{\infty}\varphi(u)^{n+1}\left[n+1+\tau_a^2\omega^2\times\frac{n(n+1)(2n+1)}{6}\right]+\Psi(u). \quad (19)$$

The sums in (19) can be performed explicitly and we obtain

$$\langle \delta x^2 \rangle = 2\sigma^2 \left[\frac{1}{u} \frac{\varphi(u)}{1-\varphi(u)} + \frac{\tau_a^2 \omega^2}{u} \left(\frac{\varphi(u)}{1-\varphi(u)} \right)^2 + \frac{2\tau_a^2 \omega^2}{u} \left(\frac{\varphi(u)}{1-\varphi(u)} \right)^3 \right] + \frac{1-\varphi(u)}{u} \quad (20)$$

and introducing $\varphi(u) = (1+u\tau)^{-1}$ the dispersion turns out to be, after inverting the Laplace transform,

$$\langle \delta x^2 \rangle = 2\frac{\sigma^2}{\tau} \left(t + \frac{\omega^2 \tau_a^2}{2\tau} t^2 + \frac{\omega^2 \tau_a^2}{3\tau^2} t^3 \right) + e^{-t/\tau}.$$

We now take the limit $\tau \to 0$, $\tau_a \to 0$, $\sigma \to 0$ keeping $D = \sigma^2/\tau$ and $A = \tau_a/\tau = 1$ constants, which we henceforth call the macroscopic limit, to get

$$\langle \Delta x^2 \rangle \equiv \lim_{\tau, \tau_a, \sigma \to 0} \langle \delta x^2 \rangle = 2D \left(t + \frac{1}{3} \omega^2 t^3 \right), \quad (21)$$

which is precisely the result obtained by solving the diffusion-advection equation [1].

2. Pure rotation

For this flow we have

$$\Omega = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix},$$

whence the equation for the random walker will be (14) with $V_k = (-\tau_a \omega k_y, \tau_a \omega k_x)$. Again we iterate this equation N times and get

$$\rho(k_x, k_y, u) = \rho(a_N k_x - b_N k_y, a_N k_y + b_N k_x, u) \times \prod_{n=0}^{N-1} \psi_0(a_n k_x - b_n k_y, a_n k_y + b_n k_x, u) + \Psi(u) \sum_{n=0}^{N-2} \prod_{m=0}^{n} \psi_0(a_m k_x - b_m k_y, a_m k_y + b_m k_x, u) + \Psi(u), \quad (22)$$

where we have defined the following real coefficients:

$$a_n = \frac{1}{2}[(1+i\omega\tau_a)^n + (1-i\omega\tau_a)^n],$$

$$b_n = \frac{1}{2i}[(1+i\omega\tau_a)^n - (1-i\omega\tau_a)^n].$$

Again when we repeat the iteration indefinitely the first summand in (22) vanishes since ρ is a bounded function and ψ_0 is everywhere smaller than 1. Now we only focus on the x direction since the problem is isotropic. We therefore average along the y direction by setting $k_y = 0$

$$\rho(k_x, 0, u) = \Psi(u) \sum_{n=0}^{\infty} \prod_{m=0}^{n} \psi_0(a_m k_x, b_m k_x, u) + \Psi(u). \quad (23)$$

We now try with the Gaussian diffusion $\psi_0(k_x, k_y, u) = \varphi(u) \exp(-\sigma^2 k_x^2 - \sigma^2 k_y^2)$ and, observing that $a_n^2 + b_n^2 = (1+\omega^2\tau_a^2)^n$, it is straightforward to obtain

$$\rho(k_x, 0, u) = \Psi(u) \left[1 + \sum_{n=1}^{\infty} \varphi(u)^n \times \exp\left(-\sigma^2 k_x^2 \frac{(1+\omega^2\tau_a^2)^n - 1}{\omega^2 \tau_a^2} \right) \right].$$

The first moment $\langle x \rangle$ vanishes and the dispersion $\langle \delta x^2 \rangle$ is again determined by $\langle x^2 \rangle$ which is easily computed and gives

$$\langle \delta x^2 \rangle = 2\sigma^2 \frac{1}{u} \frac{\varphi(u)}{1-(1+\omega^2\tau_a^2)\varphi(u)}. \quad (24)$$

We can now substitute the waiting time distribution $\varphi(u) = (1+u\tau)^{-1}$ to obtain the mean square displacement for Brownian diffusion in a circular flow (we use $D = \sigma^2/\tau$ and $A = \tau_a/\tau$)

$$\langle \delta x^2 \rangle = \frac{2D}{A\omega^2\tau_a}(e^{A\omega^2\tau_a t} - 1).$$

In the macroscopic limit $\tau_a \to 0$ we finally obtain the standard result [1]

$$\langle \Delta x^2 \rangle = 2Dt \quad (25)$$

so that diffusion remains unaffected by the rotation of the fluid.

3. Pure shear

In this kind of flow the fluid approaches the origin along one direction and separates along the perpendicular direction, if we take these two directions to be $x = y$ and $x = -y$, the flow corresponds to

$$\Omega = \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}$$

and we will have to work with Eq. (14) with $V_k = (\tau_a \omega k_y, \tau_a \omega k_x)$. We proceed as before iterating this equation N times and get

$$\rho(k_x, k_y, u) = \rho(c_N k_x + d_N k_y, c_N k_y + d_N k_x, u) \times \prod_{n=0}^{N-1} \psi_0(c_n k_x + d_n k_y, c_n k_y + d_n k_x, u) + \Psi(u) \sum_{n=0}^{N-2} \prod_{m=0}^{n} \psi_0(c_m k_x + d_m k_y, c_m k_y + d_m k_x, u) + \Psi(u), \quad (26)$$

where now the coefficients are defined as follows:

$$c_n = \frac{1}{2}[(1+\omega\tau_a)^n + (1-\omega\tau_a)^n],$$

$$d_n=\frac{1}{2}[(1+\omega\tau_a)^n-(1-\omega\tau_a)^n].$$

We just consider the x direction since the symmetry assures that the behavior along y is exactly the same, so we average along the y direction by setting $k_y=0$. When we repeat the iteration indefinitely the first summand in (26) vanishes for the same reason as in the two previous cases to yield

$$\rho(k_x,0,u)=\Psi(u)\sum_{n=0}^{\infty}\prod_{m=0}^{n}\psi_0(c_m k_x,d_m k_x,u)\ +\Psi(u). \tag{27}$$

We now introduce our particular model distribution $\psi_0(\mathbf{k},u)=\varphi(u)\ \exp(-\sigma^2k^2)$ and use $c_n^2+d_n^2=c_{2n}$ to write it as

$$\rho(k_x,0,u)=\Psi(u)\Bigg\{1+\sum_{n=1}^{\infty}\varphi(u)^n \times\exp\Bigg[-\frac{\sigma^2k_x^2}{2}\Bigg(\frac{1-(1+\omega\tau_a)^{2n}}{1-(1+\omega\tau_a)^2}+\frac{1-(1-\omega\tau_a)^{2n}}{1-(1-\omega\tau_a)^2}\Bigg)\Bigg]\Bigg\}.$$

The first moment $\langle x\rangle$ vanishes and the dispersion $\langle\delta x^2\rangle$ is again determined by $\langle x^2\rangle$ which is easily computed and gives

$$\langle\delta x^2\rangle=\sigma^2\Psi(u)\sum_{n=1}^{\infty}\varphi(u)^n\Bigg[\frac{1-(1+\omega\tau_a)^{2n}}{1-(1+\omega\tau_a)^2}+\frac{1-(1-\omega\tau_a)^{2n}}{1-(1-\omega\tau_a)^2}\Bigg]. \tag{28}$$

The summations are not difficult to carry out; we then substitute the waiting time distribution $\varphi(u)=\ (1+u\tau)^{-1}$ and with only some easy but tedious algebra we get a reasonable expression and perform its Laplace inversion to obtain the mean square displacement for Brownian diffusion in a purely sheared flow

$$\langle\delta x^2\rangle(t)=\frac{\sigma^2}{\tau}\Bigg[\frac{2\tau}{4-\omega^2\tau_a^2}-\frac{\tau^2e^{(-2\omega\tau_a+\omega^2\tau_a^2)t/\tau}}{2\omega\tau_a\tau-\omega^2\tau_a^2\tau}+\frac{\tau^2e^{(2\omega\tau_a+\omega^2\tau_a^2)t/\tau}}{2\omega\tau_a\tau+\omega^2\tau_a^2\tau}\Bigg].$$

We now take the macroscopic limit with $D=\sigma^2/\tau$ and $A=\tau_a/\tau$ constants and $\tau_a\to0$ and set $A=1$ so that we finally obtain the standard result [1]

$$\langle\Delta x^2\rangle=\frac{D}{\omega}\sinh(2\omega t). \tag{29}$$

We have therefore seen that the scheme is absolutely consistent with the standard results of diffusion theory. The novelty here is that we obtain the macroscopic mean square displacement directly from the distribution of step lengths and waiting times and we need neither the diffusion-advection equation nor the Langevin equation. This fact enables us now to attack the problem of Lévy flights of two different kinds (infinite mean square step length and infinite mean waiting time) in a sheared medium. This problem had already been addressed by starting from a postulated generalized diffusion-advection equation [15] for the case of infinite mean square step length, but the other case remained obscure because too many problems arose from the analytic manipulation of a similar *ad hoc* generalization of the macroscopic equation for that case. In contrast, the derivation along the method that we propose here is elegant and easy and even substantiates the macroscopic diffusion-advection equation which is to be applied to each case.

III. LÉVY FLIGHTS IN SHEARED MEDIUMS

We will here apply the scheme presented in the previous section to two CTRW models which in resting fluids lead to anomalous diffusion: we first present the results for the CTRW with infinite mean waiting time and we then turn to the CTRW with infinite mean square step length.

A. Infinite mean waiting time

To produce a CTRW which corresponds to a Lévy flight with an infinite mean waiting time we might choose, for instance, a probability distribution such as [21]

$$\psi_0(\mathbf{k},u)=\frac{1}{1+(u\tau)^\gamma}\exp(-\sigma^2k^2)\quad\text{with}\ \ 0<\gamma<1. \tag{30}$$

We will first try to see what the convenient generalization of the diffusion-advection equation is. To do this we recall the result (15) with $\mathbf{V_k}=\tau_a\mathbf{\Omega}^{\mathrm{T}}\cdot\mathbf{k}$, we introduce the distribution that we now propose, we invert the Fourier-Laplace transform and we take the macroscopic limit, now keeping $A=\tau_a/\tau^\gamma$ and $D=\sigma^2/\tau^\gamma$ constants as $\tau\to0$, $\tau_a\to0$ and $\sigma\to0$. This leads to the following diffusion-advection equation (we now have an A with dimensions of time to the power $1-\gamma$ and, therefore, it cannot be given the value 1 nor be absorbed within a redefinition of D and t, it must be handled as a macroscopic parameter on an equal footing with D):

$$\frac{\partial^\gamma\rho(\mathbf{x},t)}{\partial t^\gamma}+A\nabla\cdot[\mathbf{v}(\mathbf{x})\ \rho(\mathbf{x},t)]=D\nabla^2\rho(\mathbf{x},t)+\frac{t^{-\gamma}}{\Gamma(1-\gamma)}\delta(\mathbf{x}), \tag{31}$$

where we have made use of the Riemann-Liouville fractional derivative [19] much in the same spirit as in [22]. The source term appearing in Eq. (31) stems from the Laplace transform of the Riemann-Liouville derivative and incorporates the initial conditions.

Equation (31) proposes an answer to the questions arisen in some works [15,17] as to what should be the convenient generalization of the diffusion-advection equation for this type of Lévy flights. In this scheme this question is answered quite naturally and demands of the introduction of a new macroscopic parameter A which sets the right dimensions in the advection term. The nature and interpretation of this new parameter is still unclear and demands of some further investigations.

We now turn to the three types of linear shear flows, and try to see what the mean square displacement of the CTRW with long-tailed waiting time distribution is when the CTRW evolves over a linearly sheared medium.

1. Simple shear

We must use the distribution (30) in the Eq. (18) and proceed along steps analogous to the ones for Brownian diffusion. Nevertheless, we observe that the spatial part in (30) is the same as in Brownian diffusion and we can therefore borrow Eq. (20) and directly substitute our $\varphi(u)=[1+(u\tau)^{\gamma}]^{-1}$:

$$\langle \delta x^2\rangle = 2\frac{\sigma^2}{\tau^\gamma}\left(u^{-1-\gamma} + \frac{\omega^2\tau_a^2}{\tau^\gamma}u^{-1-2\gamma} + 2\frac{\omega^2\tau_a^2}{\tau^{2\gamma}}u^{-1-3\gamma}\right) + \frac{\tau^\gamma u^{-1+\gamma}}{1+(u\tau)^\gamma}.$$

Upon Laplace inversion and after the macroscopic limit with $D=\sigma^2/\tau^\gamma$ and $A=\tau_a/\tau^\gamma$ constants, we end up with

$$\langle \Delta x^2\rangle = 2D\left[\frac{1}{\Gamma(1+\gamma)}t^\gamma + \frac{2}{\Gamma(1+3\gamma)}\omega^2A^2t^{3\gamma}\right],$$

which coincides with (21) when $\gamma=1$. This is therefore the mean square displacement for this kind of Lévy flight in a simply sheared medium. This result is not easy to obtain from its associated diffusion-advection equation (31), only through this generalized CTRW scheme becomes the derivation natural.

2. Pure rotation

Again we exploit the fact that the spatial term of (30) is the same as in Brownian diffusion and just introduce $\varphi(u)=[1+(u\tau)^\gamma]^{-1}$ in Eq. (24) obtaining (remembering that for this case $D=\sigma^2/\tau^\gamma$ and $A=\tau_a/\tau^\gamma$)

$$\langle \delta x^2\rangle = 2D\frac{1}{u(u^\gamma - A\omega^2\tau_a)}.$$

We take the limit $\tau_a\to 0$ and we invert the Laplace transform to obtain the result

$$\langle \Delta x^2\rangle = \frac{2D}{\Gamma(\gamma+1)}t^\gamma,$$

which is the standard diffusion regime in a resting medium, so that for these Lévy flights, as for Brownian diffusion, the dispersion $\langle \delta x^2\rangle$ remains unaffected by the rotating fluid.

3. Pure shear

As in the previous cases we take advantage of the calculations for Brownian diffusion in a pure shear and introduce our distribution $\varphi(u)=[1+(u\tau)^\gamma]^{-1}$ into Eq. (28). We carry out the summations and we take the limit conveniently to obtain

$$\langle \Delta x^2\rangle = 2D\frac{u^{\gamma-1}}{u^{2\gamma}-4A^2\omega^2},$$

which cannot be analytically inverted but admits an asymptotic development for long times [as $u^2\to(2A\omega)^{2/\gamma}$ from the right] as

$$\frac{u^{\gamma-1}}{u^{2\gamma}-4A^2\omega^2} = \frac{(2A\omega)^{1/\gamma-1}}{24\gamma}\left[\frac{24}{u^2-(2A\omega)^{2/\gamma}} - \frac{\gamma^2-1}{(2A\omega)^{4/\gamma}}(u^2-(2A\omega)^{2/\gamma})+\cdots\right].$$

The Laplace inversion of the leading term in this expansion is now easily performed to obtain the asymptotic behavior of a Lévy flight with infinite mean waiting time in a purely sheared medium

$$\langle \Delta x^2\rangle \simeq \frac{D}{\gamma A\omega}\sinh[(2A\omega)^{1/\gamma}t]. \tag{32}$$

Comparing (32) with the mean square displacement for a Brownian walker in a purely sheared medium (29), we see that for $\gamma\to 1$ both expressions coincide and for $0<\gamma<1$ the long-time dispersion of the Lévy flight grows significantly faster for slightly sheared media $[(2\omega)^{\gamma-1}<A]$. For a more intense pure shear $[(2\omega)^{\gamma-1}>A]$, though, it is the Brownian random walker which advances more rapidly.

B. Infinite mean square step length

We now want to study CTRW's with infinite mean square step length, that is, Lévy flights of a different kind of the one considered before where we take as a step length and waiting times distribution the following:

$$\psi_0(\mathbf{k},u) = \frac{1}{1+u\tau}\exp\left[-\sigma^{2\beta}(k_x^2+k_y^2)^\beta\right] \quad \text{with} \quad 0<\beta<1. \tag{33}$$

As before we first look for the convenient generalization of the diffusion-advection equation by taking the formula (15) with $\mathbf{V_k}=\tau_a\mathbf{\Omega}^{\mathrm{T}}\cdot\mathbf{k}$ and introducing our $\psi_0(\mathbf{k},u)$. We then invert the Fourier-Laplace transform and we take the macroscopic limit, now keeping $A=\tau_a/\tau$ and $D=\sigma^{2\beta}/\tau$ constants as $\tau\to 0$, $\tau_a\to 0$, and $\sigma\to 0$. This leads to the following diffusion-advection equation (as A now has no dimensions it can be absorbed within a redefinition of D and t or be set equal to unity):

$$\frac{\partial\rho(\mathbf{x},t)}{\partial t} + \nabla\cdot[\mathbf{v}(\mathbf{x})\ \rho(\mathbf{x},t)] = D\nabla^{2\beta}\rho(\mathbf{x},t). \tag{34}$$

This is the diffusion-advection equation for Lévy flights with an infinite mean square step length. It had previously been conjectured [15,17] but this is its first analytic derivation.

We now study the typical displacement of the CTRW with long-tailed step length distribution in linearly sheared media. We consider the same three linear flows as before but we encounter now a new difficulty: the quantity that characterizes the relative diffusion of the trajectory cloud, traditionally the mean square displacement $\langle \delta x^2\rangle$, is now not well defined since the second moment of the density of particles diverges. This fact has led some researchers to discard these Lévy flights and to introduce instead the Lévy walks, where a coupling between waiting times and step lengths of the walker solves the problem (see [24] and references therein). We shall not follow this approach here. Instead, we resort to a theoretical frame which can provide us with a finite definition of the dispersion: the theory of ordered spans [18]. Following this theory, if we have a symmetric step length distribution in one dimension, the probability of having the walker within a centered interval of length m at time t, for sufficiently high m and t, is

$$p(m,t)\simeq \frac{8}{(m+1)^3}\sum_{l=0}^{\infty}\frac{d^2}{dk^2}\rho(k,t)\bigg|_{k=2\pi(l+1/2)/(m+1)}.$$

Furthermore, this theory also assures that for symmetric movements all moments of $p(m,t)$ have the same statistical properties, that is to say, all $(\langle m^\mu\rangle)^{1/\mu}$ have the same temporal dependence irrespective of the value of μ even though numerically they might be different. We will later exploit this fact by choosing conveniently μ to simplify our calculations and by defining a dispersion Δx^2 independent of μ. In the meantime we shall work with an arbitrary μ

$$\langle m^\mu\rangle(t)=\int_0^\infty m^\mu\ p(m,t)\ dm\simeq -8(1-\mu)(2\pi)^{\mu-2}\times\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)^{\mu-2}\int_0^\infty k^{-\mu}\frac{d\rho(k,t)}{dk}\,dk, \qquad (35)$$

where in the last equality we have approximated $m+1\simeq m$, we have changed variables $m=2\pi(l+1/2)/k$ as is customary in the theory of ordered spans [18], and have integrated once by parts supposing $0<\mu<1$ to keep everything finite. We will now apply this theory to each of the linear shear flows which we studied for the other cases.

1. Simple shear

We take from (18) our probability density $\rho(k,u)$ with the choice for our particular CTRW (33). This gives the probability density averaged along the y direction for this case

$$\rho(k_x,0,u)=\Psi(u)\left\{1+\sum_{n=0}^{\infty}\varphi(u)^{n+1}\times\exp\left[-\sigma^{2\beta}k_x^{2\beta}\sum_{m=0}^{n}(1+m^2\omega^2\tau_a^2)^\beta\right]\right\}$$

and we introduce it in (35) to get, after taking the derivative and performing the variable change $\xi=\sigma k[\Sigma_{m=0}^n(1+m^2\tau_a^2\omega^2)^\beta]^{1/2\beta}$,

$$\langle m^\mu\rangle(u)=C_{\mu,\beta}\sigma^\mu\ \Psi(u)\sum_{n=0}^{\infty}\varphi(u)^{n+1}\times\left[\sum_{m=0}^{n}(1+m^2\tau_a^2\omega^2)^\beta\right]^{\mu/2\beta}, \qquad (36)$$

where $C_{\mu,\beta}$ contains all the constants

$$C_{\mu,\beta}=16\beta(1-\mu)(2\pi)^{\mu-2}\times\sum_{l=0}^{\infty}\left(l+\frac{1}{2}\right)^{\mu-2}\int_0^\infty \xi^{2\beta-\mu-1}e^{-\xi^{2\beta}}d\xi.$$

$C_{\mu,\beta}$ is finite as long as $\mu<2\beta$ and $\mu\leqslant 1$ and is expressible in terms of the zeta numbers $\zeta(v)$ and the gamma function as

$$C_{\mu,\beta}=8(1-\mu)(2\pi)^{\mu-2}(2^{2-\mu}-1)\zeta(2-\mu)\Gamma\left(\frac{2\beta-\mu}{2\beta}\right). \qquad (37)$$

These constants are μ dependent so we will suppress them from our definition of Δx^2 since we are looking for a quantity which retains the main features common to all $\langle m^\mu\rangle^{1/\mu}$ but is independent of μ. We make the following definition:

$$\Delta x^2(t)\equiv 2\left(\frac{\langle M^\mu\rangle(t)}{C_{\mu,\beta}}\right)^{2/\mu}, \qquad (38)$$

where $\langle M^\mu\rangle$ stands for the macroscopic limit of $\langle m^\mu\rangle$. We will show in our examples that this definition proves to be independent of μ and coherent with the mean square displacement $\langle\Delta x^2\rangle$ of Brownian diffusion as $\beta\to 1$. Furthermore, from (36) it seems that the divergences which appear in $C_{\mu,\beta}$ and in $\langle m^\mu\rangle$ when $\mu=2\beta$ cancel out in definition (38) as we show in the next example, so that $\mu=2\beta$ is as good as any other value in order to compute Δx^2.

If the fluid is at rest, $\omega=0$, Eq. (36) simplifies to

$$\langle m^\mu\rangle(u)=C_{\mu,\beta}\sigma^\mu\ \Psi(u)\sum_{n=1}^{\infty}\varphi(u)^n n^{\mu/2\beta}. \qquad (39)$$

Now, the choice for μ which would simplify most this calculation would be $\mu=2\beta$ but this value for μ would make the constant $C_{\mu,\beta}$ (37) diverge and therefore it does not seem to be appropriate in this case. Nevertheless, since what we really seek to compute is Δx^2, if the divergences in (38) cancel out for $\mu=2\beta$ we conjecture that the Δx^2 thus computed coincides with whatever Δx^2 computed with any other μ. We corroborate this conjecture for this particular case (no shear, $\omega=0$) and then we apply it in our further developments to simplify our calculations. We take an arbitrary μ satisfying $\mu<2\beta$ and $\mu<1$ so that $C_{\mu,\beta}$ is finite. We can then write (39) as

$$\langle m^\mu\rangle(u)=C_{\mu,\beta}\sigma^\mu\ \Psi(u)\varphi(u)\ \Phi\left(\varphi(u),-\frac{\mu}{2\beta},1\right) \qquad (40)$$

in terms of the special function $\Phi(z,s,v)$ (for the definition and a list of properties see, for instance, [25]). In fact, we are interested in the behavior of $\langle m^\mu\rangle$ in the macroscopic limit, when $\varphi(u)=(1+u\tau)^{-1}\to 1$. We therefore make use of the following asymptotics for Φ:

$$\Phi(x,\nu,s)\simeq\frac{\Gamma(1-\nu)}{(1-x)^{1-\nu}}\quad \text{as}\quad x\to 1. \qquad (41)$$

So that (40) is approximated in the macroscopic limit by

$$\langle m^{\mu}\rangle(u)=C_{\mu,\beta}\sigma^{\mu}\ \Gamma\left(1+\frac{\mu}{2\beta}\right)\frac{\varphi(u)}{u(1-\varphi(u))^{\mu/2\beta}},$$

which upon substitution of the standard waiting time distribution $\varphi(u)=\ (1+u\tau)^{-1}\simeq 1-u\tau$, application of the macroscopic limit with $D=\sigma^{2\beta}/\tau$ constant and Laplace inversion yields $\langle M^{\mu}\rangle(t)$ and then substitution in (38) gives the usual scaling behavior for these Lévy flights in resting fluids (note that it is independent of μ)

$$\Delta x^2=2(Dt)^{1/\beta}. \tag{42}$$

We now show that $\mu=2\beta$ yields the same result in a much simpler way. Setting $\mu=2\beta$ in (39) transforms the previously complicated power series in a geometrical series which is straightforwardly computed and yields (formally, since $C_{2\beta,\beta}$ is known to diverge)

$$\langle m^{2\beta}\rangle(u)=C_{2\beta,\beta}\sigma^{2\beta}\ \Psi(u)\frac{\varphi(u)}{(1-\varphi(u))^2}.$$

Introducing now $\varphi(u)=\ (1+u\tau)^{-1}$, taking the macroscopic limit with $D=\sigma^{2\beta}/\tau$ and inverting the Laplace transform gives $\langle M^{2\beta}\rangle(t)$ and substitution in our definition for the dispersion Δx^2 cancels out the divergencies and leads us to exactly the same temporal dependence as we got for the case $\mu=\beta$ with finite constant $C_{\mu,\beta}$ (42)

$$\Delta x^2=2(Dt)^{1/\beta}. \tag{43}$$

The correspondence between (42) and (43) supports our conjecture that $\mu=2\beta$ is as good as any other μ moment to compute Δx^2 and that definition (38) is independent of μ.

We now study a general simple shear flow with $\omega\neq 0$: we set $\mu=2\beta$ in (36) and invert the order of the summations to get

$$\langle m^{2\beta}\rangle(u)=C_{2\beta,\beta}\sigma^{2\beta}\ \tau_a^{2\beta}\omega^{2\beta}\frac{\varphi(u)}{u}\times\sum_{m=0}^{\infty}\varphi(u)^m\left(\frac{1}{\tau_a^2\omega^2}+m^2\right)^{\beta}.$$

We will see now that, in the macroscopic limit, this summation can be calculated. First we note that the following inequalities are true:

$$\frac{\varphi}{u}\sum_{m=0}^{\infty}m^{2\beta}\varphi^m<\frac{\varphi}{u}\sum_{m=0}^{\infty}\varphi^m\left(\frac{1}{\tau_a^2\omega^2}+m^2\right)^{\beta}<\frac{\varphi}{u}\sum_{m=0}^{\infty}\varphi^m\left(\frac{1}{\tau_a\omega}+m\right)^{2\beta}.$$

From where, using the definition for $\Phi(z,s,v)$ [25] , we can write

$$\frac{\varphi^2}{u}\Phi(\varphi,-2\beta,1)<\frac{\varphi}{u}\sum_{m=0}^{\infty}\varphi^m\left(\frac{1}{\tau_a^2\omega^2}+m^2\right)^{\beta}<\frac{\varphi}{u}\Phi\left(\varphi,-2\beta,\frac{1}{\tau_a\omega}\right).$$

It is now easy to see that setting $\varphi=(1+u\tau)^{-1}$ and taking the macroscopic limit [$\tau\to 0$ with $A=\tau_a/\tau$ and $D=\sigma^{2\beta}/\tau$ constants and using the asymptotics (41)] both sides of the inequality yield the same expression, so that we can conclude

$$\langle M^{2\beta}\rangle(u)=C_{2\beta,\beta}DA^{2\beta}\omega^{2\beta}\Gamma(1+2\beta)u^{-2-2\beta}$$

and we now perform its inverse Laplace transform to get $\langle M^{2\beta}\rangle(t)$. The typical scaling behavior of a Lévy flight defined by a long-tailed step length distribution in a fluid subject to simple shear flow is then obtained by applying definition (38)

$$\Delta x^2=\frac{2}{(2\beta+1)^{1/\beta}}\omega^2D^{1/\beta}t^{2+1/\beta}, \tag{44}$$

which has exactly the same temporal dependence which was found in [15] in a heuristic manner starting from the diffusion-advection equation (34). The result (44) converges exactly to the Brownian diffusion result (21) for $\beta\to 1$ at sufficiently long times (when the formulas of the theory of ordered spans apply). It is interesting to note from (44) that Lévy flights with infinite mean square step length enhance also diffusion in a simply sheared flow with respect to the usual scaling for Brownian motion $\Delta x^2\sim t^3$ (21).

2. Pure rotation

For this other linear shear flow we proceed in a similar manner by using (23) in (35). Equation (23) yields, upon introduction of our ψ_0 (33)

$$\rho(k_x,0,u)=\Psi(u)\left[1+\sum_{n=1}^{\infty}\varphi(u)^n\times\exp\left(-\sigma^{2\beta}k_x^{2\beta}\frac{1-(1+\omega^2\tau_a^2)^{n\beta}}{1-(1+\omega^2\tau_a^2)^{\beta}}\right)\right].$$

We derive it with respect to k_x and introduce this result into formula (35). Changing variables conveniently (see the previous case) the resulting equation turns into

$$\langle m^{\mu}\rangle(u)=C_{\mu,\beta}\sigma^{\mu}\ \Psi(u)\sum_{n=1}^{\infty}\varphi(u)^n\left[\frac{1-(1+\omega^2\tau_a^2)^{n\beta}}{1-(1+\omega^2\tau_a^2)^{\beta}}\right]^{\mu/2\beta}, \tag{45}$$

where $C_{\mu,\beta}$ contains all the constants again and reads as in (37). We now choose $\mu=2\beta$ for the computation of (45) since we know that the divergences cancel out when we use definition (38) for Δx^2. With this choice, Eq. (45) simplifies enough in order to carry out the summation explicitly so that we get

$$\langle m^{2\beta}\rangle(u)=C_{2\beta,\beta}\sigma^{2\beta}\Psi(u)\times\frac{\varphi(u)}{[1-\varphi(u)][1-\varphi(u)(1+\omega^2\tau_a^2)^{\beta}]}.$$

We now introduce the waiting time distribution $\varphi(u)=(1+u\tau)^{-1}$ and take its inverse Laplace transform. We obtain, formally,

$$\langle m^{2\beta}\rangle(t)=C_{2\beta,\beta}\frac{\sigma^{2\beta}}{\tau}\,\frac{\exp\left[\frac{(1+\omega^2\tau_a^2)^\beta-1}{\tau}t\right]-1}{\frac{(1+\omega^2\tau_a^2)^\beta-1}{\tau}}.$$

We now set $D=\sigma^{2\beta}/\tau$ and $A=\tau_a/\tau$ constants and take the macroscopic limit in order to obtain the macroscopic $\langle M^{2\beta}\rangle(t)$. Subsequent substitution in definition (38) for Δx^2 then yields the macroscopic typical scaling of such a diffusive motion in a purely rotational flow

$$\Delta x^2=2(Dt)^{1/\beta},$$

which, for this CTRW as for Brownian diffusion, is invariant with respect to the characteristic behavior in a fluid at rest (43). Again for this case our definition for the dispersion Δx^2 converges exactly to the mean square displacement of Brownian diffusion $\langle\Delta x^2\rangle$ in a rotating flow (25) as $\beta\to 1$.

3. Pure shear

We follow analogous steps as in the previous two cases and introduce our distribution (33) into the formula for the probability density ρ in this kind of flow (27) to get

$$\rho(k_x,0,u)=\Psi(u)\left\{1+\sum_{n=0}^{\infty}\varphi(u)^{n+1}\exp\left[-\sigma^{2\beta}k_x^{2\beta}\sum_{m=0}^{n}\left(\frac{(1+\omega\tau_a)^{2m}+(1-\omega\tau_a)^{2m}}{2}\right)^\beta\right]\right\}.$$

Putting it into Eq. (35) and performing the variable change much in the same spirit as for (36) we end up with

$$\langle m^{\mu}\rangle(u)=C_{\mu,\beta}\sigma^{\mu}\;\Psi(u)\sum_{n=0}^{\infty}\varphi(u)^{n+1}\left[\sum_{m=0}^{n}\left(\frac{(1+\omega\tau_a)^{2m}+(1-\omega\tau_a)^{2m}}{2}\right)^\beta\right]^{\mu/2\beta}. \tag{46}$$

Again we use $\mu=2\beta$ in (46) to simplify the summations. We can then invert the order of the summations, carry out the innermost one and get

$$\langle m^{2\beta}\rangle(u)=C_{2\beta,\beta}\sigma^{2\beta}\;\frac{\Psi(u)}{1-\varphi(u)}\sum_{m=0}^{\infty}\varphi(u)^{m+1}\left(\frac{(1+\omega\tau_a)^{2m}+(1-\omega\tau_a)^{2m}}{2}\right)^\beta. \tag{47}$$

To compute the last summation we will perform some algebraic manipulations: we first take the summand $(1+\omega\tau_a)^{2m}$ out of the parentheses and then we express the resulting parentheses to the power β as a series; the summation over m is then readily carried out and we obtain

$$\langle m^{2\beta}\rangle(u)=C_{2\beta,\beta}\sigma^{2\beta}\,\frac{1}{2^\beta}\sum_{n=0}^{\infty}\binom{\beta}{n}\times\frac{\Psi(u)}{1-\varphi}\,\frac{1}{1-\varphi(1+\omega\tau_a)^{2\beta}\left(\frac{1-\omega\tau_a}{1+\omega\tau_a}\right)^{2n}}.$$

We now introduce $\varphi(u)=(1+u\tau)^{-1}$ and apply the macroscopic limit term by term keeping $D=\sigma^{2\beta}/\tau$ and $A=\tau_a/\tau=1$ constants in our infinite sum. We invert now the Laplace transform on the resulting series to get

$$\langle M^{2\beta}\rangle(t)=C_{2\beta,\beta}\frac{D}{2^\beta}\sum_{n=0}^{\infty}\binom{\beta}{n}\frac{1-e^{2\omega(\beta-2n)t}}{2\omega(2n-\beta)},$$

which is easily seen to correspond to

$$\langle M^{2\beta}\rangle(t)=\frac{C_{2\beta,\beta}}{2^\beta}D\int_0^t d\tau\sum_{n=0}^{\infty}\binom{\beta}{n}\exp[2\omega(\beta-2n)\tau]$$

or, performing the summation explicitly,

$$\langle M^{2\beta}\rangle(t)=C_{2\beta,\beta}D\int_0^t d\tau\cosh^\beta(2\omega\tau).$$

This integral can be approximated since $0<\beta<1$, for large enough t, as

$$\langle M^{2\beta}\rangle(t)\simeq C_{2\beta,\beta}\frac{D}{2\beta\omega}\cosh^{\beta-1}(2\omega t)\sinh(2\omega t)$$

and the dispersion is, therefore, according to definition (38)

$$\Delta x^2=2\left[\frac{D}{2\beta\omega}\cosh^{\beta-1}(2\omega t)\sinh(2\omega t)\right]^{1/\beta},$$

which coincides setting $\beta=1$ with the mean square displacement of Brownian diffusion in a pure shear flow (29). Asymptotically for $t\to\infty$ we then have

$$\Delta x^2\simeq\left(\frac{D}{2\beta\omega}\right)^{1/\beta}e^{2\omega t},$$

which are the same asymptotics as (29) from where we conclude that, for sufficiently long times a Lévy flight with infinite mean square step length (which traditionally leads to enhanced diffusion) does not enhance in a purely sheared flow the performance of Brownian motion. This contrasts with the previous result for Lévy flights with infinite mean waiting time which in resting fluids lead to subdiffusion but

TABLE I. Summary of the macroscopic dispersions obtained for the three kinds of flows studied (simple shear, pure rotation, and pure shear) in the three cases considered: Brownian-like diffusion, Lévy flights with infinite mean waiting time, and Lévy flights with infinite mean square step length. The symbol ∼ indicates that the result shown is a long time behavior.

(Δx^2)	Brownian	Lévy, $\langle\tau\rangle=\infty$	Lévy, $\langle l^2\rangle=\infty$
Simple shear	$2D(t+\frac{1}{3}\omega^2 t^3)$	$2D\left[\frac{1}{\Gamma(1+\gamma)}t^{\gamma}+\frac{2\omega^2A^2}{\Gamma(1+3\gamma)}t^{3\gamma}\right]$	$\sim \frac{2}{(2\beta+1)^{1/\beta}}\omega^2 D^{1/\beta}t^{2+1/\beta}$
Pure rotation	$2Dt$	$\frac{2D}{\Gamma(1+\gamma)}t^{\gamma}$	$\sim 2(Dt)^{1/\beta}$
Pure shear	$\frac{D}{\omega}\sinh(2\omega t)$	$\sim\frac{D}{\gamma A\omega}\sinh[(2A\omega)^{1/\gamma}t]$	$\sim 2\left[\frac{D}{2\beta\omega}\cosh^{\beta-1}(2\omega t)\sinh(2\omega t)\right]^{1/\beta}$

in purely sheared flows, if $(2\omega)^{\gamma-1}<A$, enhance the superdiffusion of standard diffusive movements (32).

IV. CONCLUSIONS

The main contribution of this paper is the formulation of a generalization of CTRW's to account for diffusion in moving fluids. This formulation permits to study the behavior of Lévy flights in these media in a much more rigorous way than through an *ad hoc* generalization of the diffusion-advection equation as was up to now customary. From the scheme which we put forward here the existence of a new macroscopic parameter A associated to the onset of advection appears as necessary for Lévy flights with infinite mean waiting time in order to have a well defined macroscopic limit. The nature of this parameter remains obscure but its necessity for the coherence of the scheme is strong enough to admit it before further investigations are carried out.

We have here applied this scheme to three types of two-dimensional shear flows and we have got the asymptotics for the typical rate of diffusion for Brownian walks, Lévy flights with infinite mean waiting time and Lévy flight with infinite mean square step length, respectively, which we show in Table I where with ∼ we indicate that the result is only asymptotically valid for large values of t and (Δx^2) means that in the first two columns the mean square displacement $\langle\Delta x^2\rangle$ is presented and in the last column the expressions correspond to our definition of the dispersion Δx^2 for Lévy flights with infinite mean square step lengths. We are therefore comparing different quantities but we claim that they approximately describe the same property for each diffusion mechanism. This is supported by the correspondence between the results in each row, which is satisfied at least asymptotically both for $\gamma\to 1$ and for $\beta\to 1$. It is also remarkable that in the pure rotation case the three cases coincide with the corresponding results for a resting fluid: Brownian diffusion, subdiffusion, and superdiffusion, respectively. In the simple shear case we also see that the dispersion for Brownian diffusion $\langle\Delta x^2\rangle\sim t^3$ expands faster than that for Lévy flights with infinite $\langle\tau\rangle$ and slower than the corresponding Δx^2 for Lévy flights with infinite $\langle l^2\rangle$. The astonishing result is that this trend is no longer followed in the pure shear case, where Lévy flights with infinite mean waiting time might even lead to the enhancement of the Brownian superdiffusion for sufficiently weak shears $A>(2\omega)^{\gamma-1}$. For strong enough shears, though, diffusion is again slower than in the Brownian case whereas Lévy flights with infinite mean square step length always show the same asymptotic superdiffusion as Brownian motion irrespective of the value which take β or ω.

ACKNOWLEDGMENTS

Professor David Jou and Professor José Casas-Vázquez are gratefully acknowledged for useful discussions and for their interest in this work. We also express our appreciation to Professor Y. Katayama, from College of Engineering, Nihon University, Koriyama, for drawing our attention to this field of research. The author also wishes to express his gratitude to the Department of Mathematical Physics of the University of Ulm for their hospitality and for some interesting discussions with Professor T. F. Nonnenmacher and Dr. R. Metzler, which somehow have their continuity in this work. The author is supported by the Programa de Formació d'Investigadors of the Direcció General de Recerca of the Generalitat de Catalunya. Financial support from the Dirección General de Investigación Científica y Técnica of the Spanish Ministry of Education under Grant No. PB94-0718 is acknowledged as well.

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